Q. Solve the initial-value problem

$$\dot{x} = \begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix} x, \quad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution:

The eigenvalues of $A = \begin{bmatrix} 1 & 12 \\ 3 & 1 \end{bmatrix}$ are

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 12 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)^2 - 36 = 0$

$\Rightarrow 1 - 2\lambda + \lambda^2 - 36 = 0$

$\Rightarrow \lambda^2 - 2\lambda - 35 = 0$

$\Rightarrow (\lambda - 7)(\lambda + 5) = 0$

$\lambda = 7, -5$ distinct

$\therefore$ Eigenvalues of A are $\lambda_1 = 7, \lambda_2 = -5$

for non-zero vector v,

$$[A - \lambda I] v = 0$$

$$\begin{bmatrix} 1-\lambda & 12 \\ 3 & 1-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now, eigen vector corresponding to eigenvalue $\lambda = 7$

$$\begin{bmatrix} -6 & 12 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{matrix} -6v_1 + 12v_2 = 0 \\ 3v_1 - 6v_2 = 0 \end{matrix} \Rightarrow v_1 = 2v_2$

Taking $v_2 = 1$

we get $v_1 = 2$

$\therefore$ eigenvector of A with eigenvalue 7.

$\therefore \quad x^1(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{7t}$. —— (1)

is a solution of the diff. eq^n.

Similarly, eigenvector corresponding to $\lambda_2 = -5$

$$\begin{bmatrix} 6 & 12 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{matrix} 6v_1 + 12v_2 = 0 \\ 3v_1 + 6v_2 = 0 \end{matrix} \Rightarrow v_1 = -2v_2$

Taking $v_2 = 1$, we get $v_1 = -2$

$\therefore$ eigenvector of A with eigenvalue -5

$\therefore \quad x^2(t) = \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t}$ —— (2)

is ~~a~~ second sol^n of diff. eq^n.

These solutions are linearly independent since A has distinct eigenvalues.

Hence, $x(t) = c_1 x^1(t) + c_2 x^2(t)$

$$x(t) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t} \quad \text{—— (3)}$$

where, c_1 and c_2 Constants (determined from the given initial condition)

~~Q). Solve the initial-value problem~~
~~$\dot{x} =$~~

Putting $t=0$ in eq^n (3)

$$x(0) = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}-2\\1\end{bmatrix}$$

$$\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2c_1\\c_1\end{bmatrix}+\begin{bmatrix}-2c_2\\c_2\end{bmatrix}$$

$$\Rightarrow 2c_1 - 2c_2 = 0$$
$$c_1 + c_2 = 1$$

on solving, we get

$$c_1 = \frac{1}{2}, \quad c_2 = \frac{1}{2}$$

$$\therefore x(t) = \frac{1}{2}\begin{bmatrix}2\\1\end{bmatrix}e^{7t} + \frac{1}{2}\begin{bmatrix}-2\\1\end{bmatrix}e^{-5t}$$

$$x(t) = \begin{bmatrix}e^{7t} - e^{-5t}\\ \frac{1}{2}e^{7t} + \frac{1}{2}e^{-5t}\end{bmatrix}$$

Q). Solve $\dot{x} = \begin{bmatrix}3 & 2 & 4\\2 & 0 & 2\\4 & 2 & 3\end{bmatrix}x$

Q.) Solve given initial Value Problem

i) $\dot{x} = \begin{bmatrix}1 & -3\\-2 & 2\end{bmatrix}x \quad ; \quad x(0) = \begin{bmatrix}0\\5\end{bmatrix}$

2) $\dot{x} = \begin{bmatrix}1 & -3 & 2\\0 & -1 & 0\\0 & -1 & -2\end{bmatrix}x \quad ; \quad x(0) = \begin{bmatrix}-2\\0\\3\end{bmatrix}$

Case II:
Complex eigenvalues

eg). Solve the initial Value Problem

$$\dot{x} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & -1\\0 & 1 & 1\end{bmatrix}x, \quad x(0) = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

Sol^n:

Eigenvalues of $A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & -1\\0 & 1 & 1\end{bmatrix}$ are

$$|A - \lambda I| = 0$$

$$\begin{vmatrix}1-\lambda & 0 & 0\\0 & 1-\lambda & -1\\0 & 1 & 1-\lambda\end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\{(1-\lambda)^2+1\} = 0$$

$$\Rightarrow \cancel{(1-\lambda)^3} + (\,$$

$$(1-\lambda)\ (1-2\lambda+\lambda^2+1) = 0$$

$$\Rightarrow (1-\lambda)\ (\lambda^2-2\lambda+2) = 0$$

$$\lambda = 1, \quad \lambda = \underbrace{1 \pm i}_{\text{complex}}$$

distinct

Eigenvector corresponding to eigenvalue $\lambda=1$

$[A-\lambda I]v = 0$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad v_2 = 0, \quad v_3 = 0$

$\therefore$ eigenvector corresponding to $\lambda = 1$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Hence $x^1(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t$

is one solution of given diff. eq^n

Now, eigenvector corresponding to eigenvalue $\lambda = 1+i$

$[A-\lambda I]v = 0$

$[A-(1+i)I]v = 0$

$$\begin{bmatrix} -i & 0 & 0 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -iv_1 = 0 \\ -iv_2 - v_3 = 0 \\ v_2 - iv_3 = 0 \end{cases}$$

$\Rightarrow \quad v_1 = 0$

and $v_2 = iv_3$ (from 2nd and 3rd relation)

Taking $v_3 = 1$

$\therefore$ Corresponding eigenvector is $\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$

$\therefore$ eigenvector corresponding to eigenvalue $(1+i)$ is

$$\cancel{x^2(t) =} \; e^{(1+i)t}\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$$

which is a complex valued solution

Now, $e^{(1+i)t}\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = e^t \cdot e^{it}\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$

$= e^t(\cos t + i\sin t)\begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$

$= e^t(\cos t + i\sin t)\left\{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$

$$= e^t \left[\cos t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + i\cos t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i\sin t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$= e^t \left[\begin{bmatrix} 0 \\ 0 \\ \cos t \end{bmatrix} - \begin{bmatrix} 0 \\ \sin t \\ 0 \end{bmatrix} + i \left\{ \begin{bmatrix} 0 \\ \cos t \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix} \right\} \right]$$

$$= e^t \begin{bmatrix} 0 \\ -\sin t \\ \cos t \end{bmatrix} + i e^t \begin{bmatrix} 0 \\ \cos t \\ \sin t \end{bmatrix}.$$

Lemma { from Lemma I (Page no-249)
If $x(t) = y(t) + iz(t)$ be a complex valued solution of diff. eq^n $\dot{x} = Ax$ then $y(t)$ and $z(t)$ are real valued solution of $\dot{x} = Ax$.

$\therefore$ $x^2(t) = \begin{bmatrix} 0 \\ -\sin t \\ \cos t \end{bmatrix}$ and $x^3(t) = \begin{bmatrix} 0 \\ \cos t \\ \sin t \end{bmatrix}$

are real valued solution of given diff. eq^n.

The three solutions are L.I since their initial values are

$x^1(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $x^2(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $x^3(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

$\therefore$ The complete solution is

$$x(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + c_2 e^t \begin{bmatrix} 0 \\ -\sin t \\ \cos t \end{bmatrix} + c_3 e^t \begin{bmatrix} 0 \\ \cos t \\ \sin t \end{bmatrix}$$

putting $t=0$

$$x(0) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$\Rightarrow c_1 = c_2 = c_3 = 1$

$$\therefore \quad x(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + e^t \begin{bmatrix} 0 \\ -\sin t \\ \cos t \end{bmatrix} + e^t \begin{bmatrix} 0 \\ \cos t \\ \sin t \end{bmatrix}$$

$$x(t) = e^t \begin{bmatrix} 1 \\ \cos t - \sin t \\ \cos t + \sin t \end{bmatrix}$$

Q1) Solve $\dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & -2 \\ 2 & 2 & 1 \end{bmatrix} x$

Q2) Solve the given initial value Problem

$\dot{x} = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} x \quad ; \quad x(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$